

EXISTENCE RESULTS FOR SOME DAMPED SECOND-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we make a subtle use of operator theory techniques and the well-known Schauder fixed-point principle to establish the existence of pseudo-almost automorphic solutions to some second-order damped integro-differential equations with pseudo-almost automorphic coefficients. In order to illustrate our main results, we will study the existence of pseudo-almost automorphic solutions to a structurally damped plate-like boundary value problem.

1. INTRODUCTION

Integro-differential equations play an important role when it comes to modeling various natural phenomena, see, e.g., [9, 10, 15, 27, 28, 29, 34, 43, 49, 52, 53, 57, 60, 61, 62, 71]. In recent years, noteworthy progress has been made in studying the existence of periodic, almost periodic, almost automorphic, pseudo-almost periodic, and pseudo-almost automorphic solutions to first-order integro-differential equations, see, e.g., [2, 17, 22, 23, 24, 25, 35, 36, 37, 44, 45, 60, 61, 62]. The most popular method used to deal with the existence of solutions to those first-order integro-differential equations consists of the so-called method of resolvents, see, e.g., [1, 14, 15, 36, 37, 44, 45].

Fix $\alpha \in (0, 1)$. Let \mathbb{H} be an infinite dimensional separable Hilbert space over the field of complex numbers equipped with the inner product and norm given respectively by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. The purpose of this paper consists of making use of a new approach to study the existence of pseudo-almost automorphic solutions to the class of damped second-order Volterra integro-differential equations given by

$$(1.1) \quad \frac{d^2 \varphi}{dt^2} + B \frac{d\varphi}{dt} + A\varphi = \int_{-\infty}^t C(t-s)\varphi(s)ds + f(t, \varphi),$$

where $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is an unbounded self-adjoint linear operator whose spectrum consists of isolated eigenvalues given by

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$$

as $n \rightarrow \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace, $B : D(B) \subset \mathbb{H} \mapsto \mathbb{H}$ is a positive self-adjoint linear operator such that there exist two constants $\gamma_1, \gamma_2 > 0$ and such that $\gamma_1 A^\alpha \leq B \leq \gamma_2 A^\alpha$, that is,

$$\gamma_1 \langle A^\alpha \varphi, \varphi \rangle \leq \langle B\varphi, \varphi \rangle \leq \gamma_2 \langle A^\alpha \varphi, \varphi \rangle$$

for all $\varphi \in D(B^{\frac{1}{2}}) = D(A^{\frac{\alpha}{2}})$, the mappings $C(t) : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ consist of (possibly unbounded) linear operators for each $t \in \mathbb{R}$, and the function $f : \mathbb{R} \times \mathbb{H} \mapsto \mathbb{H}$ is pseudo-almost automorphic in the first variable uniformly in the second one.

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Equations of type Eq. (1.1) arise very often in the study of natural phenomena in which a certain memory effect is taken into consideration, see, e.g., [3, 6, 46, 50, 51]. In [3, 6] for instance, equations of type Eq. (1.1) appeared in the study of a viscoelastic wave equation with memory.

The existence, uniqueness, and asymptotic behavior of solutions to Eq. (1.1) have widely been studied, see, e.g., [3, 6, 7, 8, 41, 42, 46, 50, 51, 54, 55, 56]. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to Eq. (1.1) is an untreated original problem with important applications, which constitutes the main motivation of this paper.

In this paper, we are interested in the special case $B = 2\gamma A^\alpha$ where $\gamma > 0$ is a constant, that is,

$$(1.2) \quad \frac{d^2\varphi}{dt^2} + 2\gamma A^\alpha \frac{d\varphi}{dt} + A\varphi = \int_{-\infty}^t C(t-s)\varphi(s)ds + f(t, \varphi), \quad t \in \mathbb{R}.$$

It should be mentioned that various versions of Eq. (1.2) have been investigated in the literature, see, e.g., Chen and Triggiani [11, 12], Huang [30, 31, 32, 33], and Xiao and Liang [65, 66, 67, 68, 69, 70].

Consider the polynomial Q_n^γ associated with the left hand side of Eq. (1.2), that is,

$$(1.3) \quad Q_n^\gamma(\rho) := \rho^2 + 2\gamma\lambda_n^\alpha\rho + \lambda_n$$

and denote its roots by $\rho_1^n := d_n + ie_n$ and $\rho_2^n := r_n + is_n$ for all $n \geq 1$.

In the rest of the paper, we suppose that the roots ρ_1^n and ρ_2^n satisfy: $\rho_1^n \neq \rho_2^n$ for all $n \geq 1$ and that the following crucial assumption holds: there exists $\delta_0 > 0$ such that

$$(1.4) \quad \sup_{n \geq 1} [\max(d_n, r_n)] \leq -\delta_0 < 0.$$

In order to investigate the existence of pseudo-almost automorphic solutions to Eq. (1.2), our strategy consists of rewriting it as a first-order integro-differential equation in the product space $\mathbb{B}_{\frac{1}{2}} := D(A^{\frac{1}{2}}) \times \mathbb{H}$ and then study the existence of pseudo-almost automorphic solutions to the obtained first-order integro-differential equation with the help of Schauder fixed point principle and then go back to Eq. (1.2).

Recall that the inner product of $\mathbb{B}_{\frac{1}{2}}$ is defined as follows:

$$\left(\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_{\mathbb{B}_{\frac{1}{2}}} := \langle A^{\frac{1}{2}}\varphi_1, A^{\frac{1}{2}}\psi_1 \rangle + \langle \varphi_2, \psi_2 \rangle$$

for all $\varphi_1, \psi_1 \in D(A^{\frac{1}{2}})$ and $\varphi_2, \psi_2 \in \mathbb{H}$. Its corresponding norm will be denoted $\|\cdot\|_{\mathbb{B}_{\frac{1}{2}}}$.

Letting

$$\Phi := \begin{pmatrix} \varphi \\ \varphi' \end{pmatrix} \in \mathbb{B}_{\frac{1}{2}},$$

then Eq. (1.2) can be rewritten in the following form

$$(1.5) \quad \frac{d\Phi}{dt} = \mathcal{A}\Phi + \int_{-\infty}^t C(t-s)\Phi(s)ds + F(t, \Phi(t)), \quad t \in \mathbb{R},$$

where \mathcal{A}, C are the operator matrices defined by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ -A & -A^\alpha \end{pmatrix}, \quad C = \begin{pmatrix} C \\ 0 \end{pmatrix},$$

with domain $D(\mathcal{A}) = D(A) \times [D(A^{\frac{1}{2}}) \cap D(A^\alpha)] = D(C)$ ($D(\mathcal{A}) = D(A) \times D(A^{\frac{1}{2}})$ if $0 < \alpha \leq \frac{1}{2}$ and $D(\mathcal{A}) = D(A) \times D(A^\alpha)$ if $\frac{1}{2} \leq \alpha < 1$), and the function $F : \mathbb{R} \times \mathbb{E}_{\frac{1}{2}} \mapsto \mathbb{E} := \mathbb{H} \times \mathbb{H}$ is given by

$$F(t, \Phi) = \begin{pmatrix} 0 \\ f(t, \varphi) \end{pmatrix}.$$

In order to investigate Eq. (1.5), we study the first-order differential equation in the space $\mathbb{E}_{\frac{1}{2}}$ given by,

$$(1.6) \quad \frac{d\varphi}{dt} = (\mathcal{A} + \mathfrak{B})\varphi + F(t, \varphi), \quad t \in \mathbb{R},$$

where $\mathfrak{B} : C(\mathbb{R}, D(\mathcal{A})) \mapsto \mathbb{E}_{\frac{1}{2}}$ is the linear operator defined by

$$(1.7) \quad \mathfrak{B}\varphi := \int_{-\infty}^t C(t-s)\varphi(s)ds, \quad \varphi \in C(\mathbb{R}, D(\mathcal{A}))$$

with $C(\mathbb{R}, D(\mathcal{A}))$ being the collection of all continuous functions from \mathbb{R} into $D(\mathcal{A})$.

In order to study the existence of solutions to Eq. (1.6), we will make extensive use of hyperbolic semigroup tools and fractional powers of operators, and that the linear operator \mathfrak{B} satisfies some additional assumptions. Our existence result will then be obtained through the use of the well-known Schauder fixed-point theorem. Obviously, once we establish the sought existence results for Eq. (1.6), then we can easily go back to Eq. (1.2) notably through Eq. (1.7).

The concept of pseudo almost automorphy is a powerful notion introduced in the literature by Liang *et al.* [39, 40, 63, 64]. This concept has recently generated several developments and extensions, which have been summarized in a new book by Diagana [17]. The existence of almost periodic and asymptotically almost periodic solutions to integro-differential equations of the form Eq. (1.5) in a general context has recently been established in [36, 37]. Similarly, in [45], the existence of pseudo-almost automorphic solutions to Eq. (1.5) was studied. The main method used in the above-mentioned papers are resolvents operators. However, to the best of our knowledge, the existence of pseudo-almost automorphic solutions to Eq. (1.2) is an important untreated topic with some interesting applications. Among other things, we will make extensive use of the Schauder fixed point to derive some sufficient conditions for the existence of pseudo-almost automorphic (mild) solutions to (1.6) and then to Eq. (1.2).

2. PRELIMINARIES

Some of the basic results discussed in this section are mainly taken from the following recent papers by Diagana [18, 21]. In this paper, \mathbb{H} denote an infinite dimensional separable Hilbert space over the field of complex numbers equipped with the inner product and norm given respectively by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. If A is a linear operator upon a Banach space $(\mathbb{X}, \|\cdot\|)$, then the notations $D(A)$, $\rho(A)$, $\sigma(A)$, $N(A)$, and $R(A)$ stand respectively for the domain, resolvent, spectrum, kernel, and the range of A . Similarly, if $A : D := D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is

a closed linear operator on a Banach space, one denotes its graph norm by $\|\cdot\|_D$ defined by $\|x\|_D := \|x\| + \|Ax\|$ for all $x \in D$. From the closedness of A , one can easily see that $(D, \|\cdot\|_D)$ is a Banach space. Moreover, one sets $R(\lambda, L) := (\lambda I - L)^{-1}$ for all $\lambda \in \rho(A)$. We set $Q = I - P$ for a projection P . If \mathbb{Y}, \mathbb{Z} are Banach spaces, then the space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from \mathbb{Y} into \mathbb{Z} equipped with its natural uniform operator topology $\|\cdot\|_{B(\mathbb{Y}, \mathbb{Z})}$. We also set $B(\mathbb{Y}) = B(\mathbb{Y}, \mathbb{Y})$. If $K \subset \mathbb{X}$ is a subset, we let $\overline{\text{co}} K$ denote the closed convex hull of K . Additionally, \mathbb{T} will denote the set defined by, $\mathbb{T} := \{(t, s) \in \mathbb{R} \times \mathbb{R} : t \geq s\}$. If $(\mathbb{X}, \|\cdot\|_{\mathbb{X}})$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ are Banach spaces, their product $\mathbb{X} \times \mathbb{Y} := \{(x, y) : x \in \mathbb{X}, y \in \mathbb{Y}\}$ is also a Banach when it is equipped with the norm given by

$$\|(x, y)\|_{\mathbb{X} \times \mathbb{Y}} = \sqrt{\|x\|_{\mathbb{X}}^2 + \|y\|_{\mathbb{Y}}^2} \text{ for all } (x, y) \in \mathbb{X} \times \mathbb{Y}.$$

In this paper if $\beta \geq 0$, then we set $\mathbb{E}_\beta := D(A^\beta) \times \mathbb{H}$, and $\mathbb{E} := \mathbb{H} \times \mathbb{H}$ and equip them with their corresponding topologies $\|\cdot\|_{\mathbb{E}_\beta}$ and $\|\cdot\|_{\mathbb{E}}$. Recall that $D(A^\beta)$ will be equipped with the norm defined by, $\|\varphi\|_\beta := \|A^\beta \varphi\|$ for all $\varphi \in D(A^\beta)$.

In the sequel, $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ stands for a self-adjoint (possibly unbounded) linear operator on the Hilbert space \mathbb{H} whose spectrum consists of isolated eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_n \rightarrow \infty$ with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace. Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j \geq 1}$. Clearly, for each $u \in D(A)$, where if

$$u \in D(A) := \left\{ u \in \mathbb{H} : \sum_{j=1}^{\infty} \lambda_j^2 \|E_j u\|^2 < \infty \right\}, \text{ then } Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u$$

with $E_j u = \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$. Note that $\{E_j\}_{j \geq 1}$ is a sequence of orthogonal projections on \mathbb{H} . Moreover, each $u \in \mathbb{H}$ can written as follows: $u = \sum_{j=1}^{\infty} E_j u$. It should also be mentioned that the operator $-A$ is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $u \in \mathbb{H}$,

$$S(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u$$

which in particular is exponentially stable as

$$\|S(t)\| \leq e^{-\lambda_1 t}$$

for all $t \geq 0$.

3. SECTORIAL LINEAR OPERATORS

The basic results discussed in this section are mainly taken from Diagana [17, 20].

Definition 3.1. A linear operator $B : D(B) \subset \mathbb{X} \mapsto \mathbb{X}$ (not necessarily densely defined) on a Banach space \mathbb{X} is said to be sectorial if the following hold: there exist constants $\omega \in \mathbb{R}$, $\theta \in \left(\frac{\pi}{2}, \pi\right)$, and $M > 0$ such that $\rho(B) \supset S_{\theta, \omega}$,

$$(3.1) \quad S_{\theta, \omega} := \left\{ \lambda \in \mathbb{C} : \lambda \neq \omega, \quad |\arg(\lambda - \omega)| < \theta \right\}, \text{ and}$$

$$(3.2) \quad \|R(\lambda, B)\| \leq \frac{M}{|\lambda - \omega|}, \quad \lambda \in S_{\theta, \omega}.$$

Example 3.2. Let $p \geq 1$ and let $\Omega \subset \mathbb{R}^d$ be open bounded subset with C^2 boundary $\partial\Omega$. Let $\mathbb{X} := L^p(\Omega)$ be the Lebesgue space equipped with the norm, $\|\cdot\|_p$ defined by,

$$\|\varphi\|_p = \left(\int_{\Omega} |\varphi(x)|^p dx \right)^{1/p}.$$

Define the operator A as follows:

$$D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad B(\varphi) = \Delta\varphi, \quad \forall \varphi \in D(B),$$

where $\Delta = \sum_{k=1}^d \frac{\partial^2}{\partial x_k^2}$ is the Laplace operator. It can be checked that the operator B is sectorial on $L^p(\Omega)$.

It is well-known [47] that if $B : D(B) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator, then it generates an analytic semigroup $(T(t))_{t \geq 0}$, which maps $(0, \infty)$ into $B(\mathbb{X})$ and such that there exist $M_0, M_1 > 0$ with

$$(3.3) \quad \|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0,$$

$$(3.4) \quad \|t(A - \omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0.$$

In this paper, we suppose that the semigroup $(T(t))_{t \geq 0}$ is hyperbolic, that is, there exist a projection P and constants $M, \delta > 0$ such that $T(t)$ commutes with P , $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and the following hold

$$(3.5) \quad \|T(t)Px\| \leq M e^{-\delta t} \|x\| \quad \text{for } t \geq 0,$$

$$(3.6) \quad \|T(t)Qx\| \leq M e^{\delta t} \|x\| \quad \text{for } t \leq 0,$$

where $Q := I - P$ and, for $t \leq 0$, $T(t) := (T(-t))^{-1}$.

Recall that the analytic semigroup $(T(t))_{t \geq 0}$ associated with B is hyperbolic if and only if $\sigma(B) \cap i\mathbb{R} = \emptyset$, see details in [26, Proposition 1.15, pp.305].

Definition 3.3. Let $\alpha \in (0, 1)$. A Banach space $(\mathbb{X}_\alpha, \|\cdot\|_\alpha)$ is said to be an intermediate space between $D(B)$ and \mathbb{X} , or a space of class \mathcal{J}_α , if $D(B) \subset \mathbb{X}_\alpha \subset \mathbb{X}$ and there is a constant $c > 0$ such that

$$(3.7) \quad \|x\|_\alpha \leq c \|x\|^{1-\alpha} \|x\|_B^\alpha, \quad x \in D(B).$$

Concrete examples of \mathbb{X}_α include $D((-B^\alpha))$ for $\alpha \in (0, 1)$, the domains of the fractional powers of B , the real interpolation spaces $D_B(\alpha, \infty)$, $\alpha \in (0, 1)$, defined as the space of all $x \in \mathbb{X}$ such that,

$$[x]_\alpha = \sup_{0 < t \leq 1} \|t^{1-\alpha} B T(t)x\| < \infty$$

with the norm

$$\|x\|_\alpha = \|x\| + [x]_\alpha,$$

the abstract Hölder spaces $D_B(\alpha) := \overline{D(B)}^{\|\cdot\|_\alpha}$ as well as the complex interpolation spaces $[\mathbb{X}, D(B)]_\alpha$.

For a hyperbolic analytic semigroup $(T(t))_{t \geq 0}$, one can easily check that similar estimations as both Eq. (3.5) and Eq. (3.6) still hold with the α -norms $\|\cdot\|_\alpha$. In fact, as the part of A in $R(Q)$ is bounded, it follows from Eq. (3.6) that

$$\|B T(t) Q x\| \leq C' e^{\delta t} \|x\| \quad \text{for } t \leq 0.$$

Hence, from Eq. (3.7) there exists a constant $c(\alpha) > 0$ such that

$$(3.8) \quad \|T(t) Q x\|_\alpha \leq c(\alpha) e^{\delta t} \|x\| \quad \text{for } t \leq 0.$$

In addition to the above, the following holds

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathbb{X}, \mathbb{X}_\alpha)} \|T(t-1)Px\|, \quad t \geq 1,$$

and hence from Eq. (3.5), one obtains

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t} \|x\|, \quad t \geq 1,$$

where M' depends on α . For $t \in (0, 1]$, by Eq. (3.4) and Eq. (3.7),

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha} \|x\|.$$

Hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$(3.9) \quad \|T(t)Px\|_\alpha \leq M(\alpha) t^{-\alpha} e^{-\gamma t} \|x\| \quad \text{for } t > 0.$$

Remark 3.4. Note that if the analytic semigroup $T(t)$ is exponential stable, that is, there exists constants $N, \delta > 0$ such that $\|T(t)\| \leq N e^{-\delta t}$ for $t \geq 0$, then the projection $P = I$ ($Q = I - P(t) = 0$). In that case, Eq. (3.9) still holds and can be rewritten as follows: for all $x \in \mathbb{X}$,

$$(3.10) \quad \|T(t)x\|_\alpha \leq M(\alpha) e^{-\frac{\gamma}{2}t} t^{-\alpha} \|x\|.$$

For more on interpolation spaces and related issues, we refer the reader to the following excellent books Amann [5] and Lunardi [47].

3.1. Pseudo-Almost Automorphic Functions. Let $BC(\mathbb{R}, \mathbb{X})$ stand for the Banach space of all bounded continuous functions $\varphi : \mathbb{R} \mapsto \mathbb{X}$, which we equip with the sup-norm defined by $\|\varphi\|_\infty := \sup_{t \in \mathbb{R}} \|\varphi(t)\|$ for all $\varphi \in BC(\mathbb{R}, \mathbb{X})$. If $\beta \geq 0$, we will also be using the following notions,

$$\|\Phi\|_{\mathbb{E}_{\beta, \infty}} := \sup_{t \in \mathbb{R}} \|\Phi(t)\|_{\mathbb{E}_\beta}$$

for $\Phi \in BC(\mathbb{R}, \mathbb{E}_\beta)$, and

$$\|\varphi\|_{\beta, \infty} := \sup_{t \in \mathbb{R}} \|\varphi(t)\|_\beta$$

for $\varphi \in BC(\mathbb{R}, D(A^\beta))$.

Definition 3.5. [17] A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$g(t) := \lim_{n \rightarrow \infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} g(t - s_n) = f(t)$$

for each $t \in \mathbb{R}$.

If the convergence above is uniform in $t \in \mathbb{R}$, then f is almost periodic in the classical Bochner's sense. Denote by $AA(\mathbb{X})$ the collection of all almost automorphic functions $\mathbb{R} \mapsto \mathbb{X}$. Note that $AA(\mathbb{X})$ equipped with the sup-norm turns out to be a Banach space.

Among other things, almost automorphic functions satisfy the following properties.

Theorem 3.6. [17] If $f, f_1, f_2 \in AA(\mathbb{X})$, then

- (i) $f_1 + f_2 \in AA(\mathbb{X})$,
- (ii) $\lambda f \in AA(\mathbb{X})$ for any scalar λ ,
- (iii) $f_\alpha \in AA(\mathbb{X})$ where $f_\alpha : \mathbb{R} \rightarrow \mathbb{X}$ is defined by $f_\alpha(\cdot) = f(\cdot + \alpha)$,
- (iv) the range $\mathcal{R}_f := \{f(t) : t \in \mathbb{R}\}$ is relatively compact in \mathbb{X} , thus f is bounded in norm,

(v) if $f_n \rightarrow f$ uniformly on \mathbb{R} where each $f_n \in AA(\mathbb{X})$, then $f \in AA(\mathbb{X})$ too.

Definition 3.7. Let \mathbb{Y} be another Banach space. A jointly continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ is said to be almost automorphic in $t \in \mathbb{R}$ if $t \mapsto F(t, x)$ is almost automorphic for all $x \in K$ ($K \subset \mathbb{Y}$ being any bounded subset). Equivalently, for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that

$$G(t, x) := \lim_{n \rightarrow \infty} F(t + s_n, x)$$

is well defined in $t \in \mathbb{R}$ and for each $x \in K$, and

$$\lim_{n \rightarrow \infty} G(t - s_n, x) = F(t, x)$$

for all $t \in \mathbb{R}$ and $x \in K$.

The collection of such functions will be denoted by $AA(\mathbb{R} \times \mathbb{X})$.

For more on almost automorphic functions and their generalizations, we refer the reader to the recent book by Diagana [17].

Define (see Diagana [17, 19]) the space $PAP_0(\mathbb{R}, \mathbb{X})$ as the collection of all functions $\varphi \in BC(\mathbb{R}, \mathbb{X})$ satisfying,

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|\varphi(s)\| ds = 0.$$

Similarly, $PAP_0(\mathbb{R} \times \mathbb{X})$ will denote the collection of all bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ such that

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T \|F(s, x)\| ds = 0$$

uniformly in $x \in K$, where $K \subset \mathbb{Y}$ is any bounded subset.

Definition 3.8. (Liang *et al.* [39] and Xiao *et al.* [63]) A function $f \in BC(\mathbb{R}, \mathbb{X})$ is called pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in PAP_0(\mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{X})$.

The functions g and ϕ appearing in Definition 3.8 are respectively called the *almost automorphic* and the *ergodic perturbation* components of f .

Definition 3.9. Let \mathbb{Y} be another Banach space. A bounded continuous function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $AA(\mathbb{R} \times \mathbb{X})$ whenever it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{R} \times \mathbb{X})$ and $\Phi \in PAP_0(\mathbb{R} \times \mathbb{X})$. The collection of such functions will be denoted by $PAA(\mathbb{R} \times \mathbb{X})$.

A substantial result is the next theorem, which is due to Xiao *et al.* [63].

Theorem 3.10. [63] *The space $PAA(\mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space.*

Theorem 3.11. [63] *If \mathbb{Y} is another Banach space, $f : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $PAA(\mathbb{R} \times \mathbb{X})$ and if $x \mapsto f(t, x)$ is uniformly continuous on each bounded subset K of \mathbb{Y} uniformly in $t \in \mathbb{R}$, then the function defined by $h(t) = f(t, \varphi(t))$ belongs to $PAA(\mathbb{X})$ provided $\varphi \in PAA(\mathbb{Y})$.*

For more on pseudo-almost automorphic functions and their generalizations, we refer the reader to the recent book by Diagana [17].

4. MAIN RESULTS

Fix $\beta \in (0, 1)$. Consider the first-order differential equations,

$$(4.1) \quad \frac{d\varphi}{dt} = A\varphi + g(t), \quad t \in \mathbb{R},$$

and

$$(4.2) \quad \frac{d\varphi}{dt} = (A + B)\varphi + f(t, \varphi), \quad t \in \mathbb{R},$$

where $A : D(A) \subset \mathbb{X} \mapsto \mathbb{X}$ is a sectorial linear operator on a Banach space \mathbb{X} , $B : C(\mathbb{R}, D(A)) \mapsto \mathbb{X}$ is a linear operator, and $g : \mathbb{R} \mapsto \mathbb{X}$ and $f : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}$ are bounded continuous functions.

To study the existence of pseudo-almost automorphic mild solutions to Eq. (4.1) (and hence Eq. (4.2)), we will need the following assumptions,

- (H.1) The linear operator A is sectorial. Moreover, if $T(t)$ denotes the analytic semigroup associated with it, we suppose that $T(t)$ is hyperbolic, that is,

$$\sigma(A) \cap i\mathbb{R} = \emptyset.$$

- (H.2) The semigroup $T(t)$ is not only compact for $t > 0$ but also is exponentially stable, i.e., there exists constants $N, \delta > 0$ such that

$$\|T(t)\| \leq Ne^{-\delta t}$$

for $t \geq 0$.

- (H.3) The linear operator $B : BC(\mathbb{R}, \mathbb{X}_\beta) \mapsto \mathbb{X}$, where $\mathbb{X}_\beta := D((-A)^\beta)$, is bounded. Moreover, the following holds,

$$C_0 := \|B\|_{B(BC(\mathbb{R}, \mathbb{X}_\beta), \mathbb{X})} \leq \frac{1}{2d(\beta)},$$

where $d(\beta) := M(\beta)(2\delta^{-1})^{1-\beta}\Gamma(1-\beta)$.

- (H.4) The function $f : \mathbb{R} \times \mathbb{X}_\beta \mapsto \mathbb{X}$ is pseudo-almost automorphic in the first variable uniformly in the second one. For each bounded subset $K \subset \mathbb{X}_\beta$, $f(\mathbb{R}, K)$ is bounded. Moreover, the function $u \mapsto f(t, u)$ is uniformly continuous on any bounded subset K of \mathbb{X}_β for each $t \in \mathbb{R}$. Finally, we suppose that there exists $L > 0$ such that

$$\sup_{t \in \mathbb{R}, \|\varphi\|_\beta \leq L} \|f(t, \varphi)\| \leq \frac{L}{2d(\beta)}.$$

- (H.5) If $(u_n)_{n \in \mathbb{N}} \subset PAA(\mathbb{X}_\beta)$ is uniformly bounded and uniformly convergent upon every compact subset of \mathbb{R} , then $f(\cdot, u_n(\cdot))$ is relatively compact in $BC(\mathbb{R}, \mathbb{X})$.

Remark 4.1. Note that if (H.3) holds, then it can be easily shown that the linear operator B maps $PAA(\mathbb{X}_\beta)$ into $PAA(\mathbb{X})$.

Definition 4.2. Under assumption (H.1), a continuous function $\varphi : \mathbb{R} \mapsto \mathbb{X}$ is said to be a mild solution to Eq. (4.1) provided that

$$(4.3) \quad \varphi(t) = T(t-s)\varphi(s) + \int_s^t T(t-\tau)g(\tau)d\tau, \quad \forall (t, s) \in \mathbb{T}.$$

Lemma 4.3. [17] Suppose assumptions (H.1)–(H.2) hold. If $g : \mathbb{R} \mapsto \mathbb{X}$ is a bounded continuous function, then φ given by

$$(4.4) \quad \varphi(t) := \int_{-\infty}^t T(t-s)g(s)ds$$

for all $t \in \mathbb{R}$, is the unique bounded mild solution to Eq. (4.1).

Definition 4.4. Under assumptions (H.1), (H.2), and (H.3) and if $f : \mathbb{R} \times \mathbb{X}_\beta \mapsto \mathbb{X}$ is a bounded continuous function, then a continuous function $\varphi : \mathbb{R} \mapsto \mathbb{X}_\beta$ satisfying

$$(4.5) \quad \varphi(t) = T(t-s)\varphi(s) + \int_s^t T(t-s)[B\varphi(s) + f(s, \varphi(s))]ds, \quad \forall (t, s) \in \mathbb{T}$$

is called a mild solution to Eq. (4.2).

Under assumptions (H.1), (H.2), and (H.3) and if $f : \mathbb{R} \times \mathbb{X}_\beta \mapsto \mathbb{X}$ is a bounded continuous function, it can be shown that the function $\varphi : \mathbb{R} \mapsto \mathbb{X}_\beta$ defined by

$$(4.6) \quad \varphi(t) = \int_{-\infty}^t T(t-s)[B\varphi(s) + f(s, \varphi(s))]ds$$

for all $t \in \mathbb{R}$, is a mild solution to Eq. (4.2).

Define the following integral operator,

$$(S\varphi)(t) = \int_{-\infty}^t T(t-s)[B\varphi(s) + f(s, \varphi(s))]ds.$$

We have

Lemma 4.5. Under assumptions (H.1)–(H.2)–(H.3) and if $f : \mathbb{R} \times \mathbb{X}_\beta \mapsto \mathbb{X}$ is a bounded continuous function, then the mapping $S : BC(\mathbb{R}, \mathbb{X}_\beta) \mapsto BC(\mathbb{R}, \mathbb{X}_\beta)$ is well-defined and continuous.

Proof. We first show that S is well-defined and that $S(BC(\mathbb{R}, \mathbb{X}_\beta)) \subset BC(\mathbb{R}, \mathbb{X}_\beta)$. Indeed, letting $u \in BC(\mathbb{R}, \mathbb{X}_\beta)$, $g(t) := f(t, u(t))$, and using Eq. (3.10), we obtain

$$\begin{aligned} \|Su(t)\|_\beta &\leq \int_{-\infty}^t \|T(t-s)[Bu(s) + g(s)]\|_\beta ds \\ &\leq \int_{-\infty}^t M(\beta)e^{-\frac{\delta}{2}(t-s)}(t-s)^{1-\beta}[\|Bu(s)\| + \|g(s)\|]ds \\ &\leq \int_{-\infty}^t M(\beta)e^{-\frac{\delta}{2}(t-s)}(t-s)^{1-\beta}[C_0\|u(s)\|_\beta + \|g(s)\|]ds \\ &\leq d(\beta)(C_0\|u\|_{\beta, \infty} + \|g\|_\infty), \end{aligned}$$

for all $t \in \mathbb{R}$, where $d := M(\beta)(2\delta^{-1})^{1-\beta}\Gamma(1-\beta)$, and hence $Su : \mathbb{R} \mapsto \mathbb{X}_\beta$ is bounded.

To complete the proof it remains to show that S is continuous. For that, set

$$F(s, u(s)) := Bu(s) + g(s) = Bu(s) + f(s, u(s)), \quad \forall s \in \mathbb{R}.$$

Consider an arbitrary sequence of functions $u_n \in BC(\mathbb{R}, \mathbb{X}_\beta)$ that converges uniformly to some $u \in BC(\mathbb{R}, \mathbb{X}_\beta)$, that is, $\|u_n - u\|_{\beta, \infty} \rightarrow 0$ as $n \rightarrow \infty$.

Now

$$\begin{aligned}
\|Su(t) - Su_n(t)\|_\beta &= \left\| \int_{-\infty}^t T(t-s)[F(s, u_n(s)) - F(s, u(s))] ds \right\|_\beta \\
&\leq M(\beta) \int_{-\infty}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|F(s, u_n(s)) - F(s, u(s))\| ds \\
&\leq M(\beta) \int_{-\infty}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|f(s, u_n(s)) - f(s, u(s))\| ds \\
&\quad + M(\beta) \int_{-\infty}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|B(u_n(s) - u(s))\| ds \\
&\leq M(\beta) \int_{-\infty}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|f(s, u_n(s)) - f(s, u(s))\| ds \\
&\quad + d(\beta)C_0 \|u_n - u\|_{\beta, \infty}.
\end{aligned}$$

Using the continuity of the function $f : \mathbb{R} \times \mathbb{X}_\beta \mapsto \mathbb{X}$ and the Lebesgue Dominated Convergence Theorem we conclude that

$$\left\| \int_{-\infty}^t T(t-s)P(s)[f(s, u_n(s)) - f(s, u(s))] ds \right\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore, $\|Su_n - Su\|_{\beta, \infty} \rightarrow 0$ as $n \rightarrow \infty$. The proof is complete. \square

Lemma 4.6. *Under assumptions (H.1)—(H.4), then $S(PAA(\mathbb{X}_\beta)) \subset PAA(\mathbb{X}_\beta)$.*

Proof. Let $u \in PAA(\mathbb{X}_\beta)$ and define $h(s) := f(s, u(s)) + Bu(s)$ for all $s \in \mathbb{R}$. Using (H.4) and Theorem 3.11 it follows that the function $s \mapsto f(s, u(s))$ belongs to $PAA(\mathbb{X})$. Similarly, using Remark 4.1 it follows that the function $s \mapsto Bu(s)$ belongs to $PAA(\mathbb{X})$. In view of the above, the function $s \mapsto h(s)$ belongs to $PAA(\mathbb{X})$.

Now write $h = h_1 + h_2 \in PAA(\mathbb{X})$ where $h_1 \in AA(\mathbb{X})$ and $h_2 \in PAP_0(\mathbb{X})$ and set

$$Rh_j(t) := \int_{-\infty}^t T(t-s)h_j(s)ds \text{ for all } t \in \mathbb{R}, \quad j = 1, 2.$$

Our first task consists of showing that $R(AA(\mathbb{X})) \subset AA(\mathbb{X}_\beta)$. Indeed, using the fact that $h_1 \in AA(\mathbb{X})$, for every sequence of real numbers $(\tau'_n)_{n \in \mathbb{N}}$ there exist a subsequence $(\tau_n)_{n \in \mathbb{N}}$ and a function f_1 such that

$$f_1(t) := \lim_{n \rightarrow \infty} h_1(t + \tau_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} f_1(t - \tau_n) = h_1(t)$$

for each $t \in \mathbb{R}$.

Now

$$\begin{aligned}
(Rh_1)(t + \tau_n) - (Rf_1)(t) &= \int_{-\infty}^{t+\tau_n} T(t+\tau_n-s)h_1(s)ds - \int_{-\infty}^t T(t-s)f_1(s)ds \\
&= \int_{-\infty}^t T(t-s)h_1(s+\tau_n)ds - \int_{-\infty}^t T(t-s)f_1(s)ds \\
&= \int_{-\infty}^t T(t-s)(h_1(s+\tau_n) - f_1(s))ds.
\end{aligned}$$

From Eq. (3.10) and the Lebesgue Dominated Convergence Theorem, it easily follows that

$$\begin{aligned} \left\| \int_{-\infty}^t T(t-s)(h_1(s+\tau_n) - f_1(s))ds \right\|_{\beta} &\leq \int_{-\infty}^t \left\| T(t-s)(h_1(s+\tau_n) - f_1(s)) \right\|_{\beta} ds \\ &\leq M(\beta) \int_{-\infty}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|h_1(s+\tau_n) - f_1(s)\| ds \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

and hence

$$(Rf_1)(t) = \lim_{n \rightarrow \infty} (Rh_1)(t + \tau_n)$$

for all $t \in \mathbb{R}$.

Using similar arguments as above one obtains that

$$(Rh_1)(t) = \lim_{n \rightarrow \infty} (Rf_1)(t - \tau_n)$$

for all $t \in \mathbb{R}$, which yields, $t \mapsto (Sh_1)(t)$ belongs to $AA(\mathbb{X}_{\beta})$.

The next step consists of showing that $R(PAP_0(\mathbb{X})) \subset PAP_0(\mathbb{X}_{\beta})$. Obviously, $Rh_2 \in BC(\mathbb{R}, \mathbb{X}_{\beta})$ (see Lemma 4.5). Using the fact that $h_2 \in PAP_0(\mathbb{X})$ and Eq. (3.10) it can be easily shown that $(Rh_2) \in PAP_0(\mathbb{X}_{\beta})$. Indeed, for $r > 0$,

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \left\| \int_{-\infty}^t T(t-s)h_2(s)ds \right\|_{\beta} dt &\leq \frac{M(\beta)}{2r} \int_{-r}^r \int_0^{\infty} e^{\frac{\delta}{2}s} s^{-\beta} \|h_2(t-s)\| ds dt \\ &\leq M(\beta) \int_0^{\infty} e^{\frac{\delta}{2}s} s^{-\beta} \left(\frac{1}{2r} \int_{-r}^r \|h_2(t-s)\| dt \right) ds. \end{aligned}$$

Using the fact that $PAP_0(\mathbb{X})$ is translation-invariant it follows that

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \|h_2(t-s)\| dt = 0,$$

as $t \mapsto h_2(t-s) \in PAP_0(\mathbb{X})$ for every $s \in \mathbb{R}$.

One completes the proof by using the Lebesgue Dominated Convergence Theorem. In summary, $(Rh_2) \in PAP_0(\mathbb{X}_{\beta})$, which completes the proof. \square

Theorem 4.7. *Suppose assumptions (H.1)—(H.5) hold, then Eq. (4.2) has at least one pseudo-almost automorphic mild solution*

Proof. Let $B_{\beta} = \{u \in PAA(\mathbb{X}_{\beta}) : \|u\|_{\beta} \leq L\}$. Using the proof of Lemma 4.5 it follows that B_{β} is a convex and closed set. Now using Lemma 4.6 it follows that $S(B_{\beta}) \subset PAA(\mathbb{X}_{\beta})$.

Now for all $u \in B_{\beta}$,

$$\begin{aligned} \|Su(t)\|_{\beta} &\leq \int_{-\infty}^t \|T(t-s)[Bu(s) + g(s)]\|_{\beta} ds \\ &\leq \int_{-\infty}^t M(\beta) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\beta} [\|Bu(s)\| + \|f(s, u(s))\|] ds \\ &\leq \int_{-\infty}^t M(\beta) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\beta} [C_0 \|u(s)\|_{\beta} + \|f(s, u(s))\|] ds \\ &\leq d(\beta) \left(\frac{L}{2d(\beta)} + \frac{L}{2d(\beta)} \right) \\ &= L \end{aligned}$$

for all $t \in \mathbb{R}$, and hence $Su \in B_{\beta}$.

To complete the proof, we have to prove the following:

- a) That $V = \{Su(t) : u \in B_\beta\}$ is a relatively compact subset of \mathbb{X}_β for each $t \in \mathbb{R}$;
- b) That $W = \{Su : u \in B_\beta\} \subset BC(\mathbb{R}, \mathbb{X}_\beta)$ is equi-continuous.

To show a), fix $t \in \mathbb{R}$ and consider an arbitrary $\varepsilon > 0$.

Now

$$\begin{aligned} (S_\varepsilon u)(t) &:= \int_{-\infty}^{t-\varepsilon} T(t-s)F(s, u(s))ds, \quad u \in B_\beta \\ &= T(\varepsilon) \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)F(s, u(s))ds, \quad u \in B_\beta \\ &= T(\varepsilon)(Su)(t-\varepsilon), \quad u \in B_\beta \end{aligned}$$

and hence $V_\varepsilon := \{S_\varepsilon u(t) : u \in B_\beta\}$ is relatively compact in \mathbb{X}_β as the evolution family $T(\varepsilon)$ is compact by assumption.

Now

$$\begin{aligned} &\|Su(t) - T(\varepsilon) \int_{-\infty}^{t-\varepsilon} T(t-\varepsilon-s)F(s, u(s))ds\|_\beta \\ &\leq \int_{t-\varepsilon}^t \|T(t-s)F(s, u(s))\|_\beta ds \\ &\leq M(\beta) \int_{t-\varepsilon}^t e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\beta} \|F(s, u(s))\| ds \\ &\leq M(\beta) \int_{t-\varepsilon}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|g(s)\| ds + M(\beta) \int_{t-\varepsilon}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|Bu(s)\| ds \\ &\leq M(\beta) \int_{t-\varepsilon}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} \|f(s, u(s))\| ds + M(\beta)C_0\|u\|_{\beta, \infty} \int_{t-\varepsilon}^t (t-s)^{-\beta} e^{-\frac{\delta}{2}(t-s)} ds \\ &\leq M(\beta)L(d^{-1}(\beta) + C_0) \int_{t-\varepsilon}^t (t-s)^{-\alpha} ds \\ &= M(\beta)L(d^{-1}(\beta) + C_0)\varepsilon^{1-\beta}(1-\beta)^{-1}, \end{aligned}$$

and hence the set $V := \{Su(t) : u \in B_\beta\} \subset \mathbb{X}_\beta$ is relatively compact.

The proof for b) follows along the same lines as in Li *et al.* [38, Theorem 31] and hence is omitted.

The rest of the proof slightly follows along the same lines as in Diagana [18]. Indeed, since B_β is a closed convex subset of $PAA(\mathbb{X}_\beta)$ and that $S(B_\beta) \subset B_\beta$, it follows that $\overline{co} S(B_\beta) \subset B_\beta$. Consequently,

$$S(\overline{co} S(B_\beta)) \subset S(B_\beta) \subset \overline{co} S(B_\beta).$$

Further, it is not hard to see that $\{u(t) : u \in \overline{co} S(B_\beta)\}$ is relatively compact in \mathbb{X}_β for each fixed $t \in \mathbb{R}$ and that functions in $\overline{co} S(B_\beta)$ are equi-continuous on \mathbb{R} . Using Arzelà-Ascoli theorem, we deduce that the restriction of $\overline{co} S(B_\beta)$ to any compact subset I of \mathbb{R} is relatively compact in $C(I, \mathbb{X}_\beta)$.

In summary, $S : \overline{co} S(B_\beta) \mapsto \overline{co} S(B_\beta)$ is continuous and compact. Using the Schauder fixed point it follows that S has a fixed-point, which obviously is a pseudo-almost automorphic mild solution to Eq. (4.2).

□

Fix $\alpha \in [\frac{1}{2}, 1)$. In order to study Eq. (1.6), we let $\beta = \frac{1}{2}$ and suppose that the following additional assumption holds:

(H.6) There exists a function $\rho \in L^1(\mathbb{R}, (0, \infty))$ with $\|\rho\|_{L^1(\mathbb{R}, (0, \infty))} \leq \frac{1}{2d(\frac{1}{2})}$ such that

$$\|C(t)\varphi\|_{\mathbb{B}} \leq \rho(t)\|\varphi\|_{\mathbb{B}_{\frac{1}{2}}}$$

for all $\varphi \in \mathbb{B}_{\frac{1}{2}}$ and $t \in \mathbb{R}$.

Corollary 4.8. *Under assumptions (H.1)–(H.2)–(H.4)–(H.5)–(H.6), then Eq. (1.6) (and hence Eq. (1.5) and Eq. (1.2)) has at least one pseudo-almost automorphic mild solution.*

Proof. It suffices to show that \mathcal{A} and \mathcal{B} satisfy similar assumptions as (H.1)–(H.2)–(H.3) and that F satisfies similar assumptions as (H.4)–(H.5).

Step 1. Assumption (H.6) yields \mathcal{B} satisfies similar assumption as (H.3), where \mathcal{B} is defined by

$$\mathcal{B}\varphi(t) := \int_{-\infty}^t C(t-s)\varphi(s)ds.$$

Indeed, since the function ρ is integrable, it is clear that the operator \mathcal{B} belong to $B(BC(\mathbb{R}, \mathbb{B}_{\frac{1}{2}}), \mathbb{B})$ with $\|\mathcal{B}\|_{B(BC(\mathbb{R}, \mathbb{B}_{\frac{1}{2}}), \mathbb{B})} \leq \|\rho\|_{L^1(\mathbb{R}, (0, \infty))}$. In fact, we take $C_0 = \|\rho\|_{L^1(\mathbb{R}, (0, \infty))}$. The fact the function $t \mapsto \mathcal{B}\varphi(t)$ is pseudo-almost automorphic for any $\varphi \in PAA(\mathbb{B}_{\frac{1}{2}})$ is guaranteed by Remark 4.1. However, for the sake of clarity, we will show it. Indeed, write $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1 \in AA(\mathbb{B}_{\frac{1}{2}})$ and $\varphi_2 \in PAP_0(\mathbb{B}_{\frac{1}{2}})$. Using the fact that the function $t \mapsto \varphi_1(t)$ belongs to $AA(\mathbb{B}_{\frac{1}{2}})$, for every sequence of real numbers $(\tau'_n)_{n \in \mathbb{N}}$ there exist a subsequence $(\tau_n)_{n \in \mathbb{N}}$ and a function ψ_1 such that

$$\psi_1(t) := \lim_{n \rightarrow \infty} \varphi_1(t + \tau_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \rightarrow \infty} \psi_1(t - \tau_n) = \varphi_1(t)$$

for each $t \in \mathbb{R}$.

Now

$$\begin{aligned} \mathcal{B}\varphi_1(t + \tau_n) - \mathcal{B}\psi_1(t) &= \int_{-\infty}^{t+\tau_n} C(t + \tau_n - s)\varphi_1(s)ds - \int_{-\infty}^t C(t - s)\psi_1(s)ds \\ &= \int_{-\infty}^t C(t - s)\varphi_1(s + \tau_n)ds - \int_{-\infty}^t C(t - s)\psi_1(s)ds \\ &= \int_{-\infty}^t C(t - s)(\varphi_1(s + \tau_n) - \psi_1(s))ds \end{aligned}$$

and hence

$$\begin{aligned} \|\mathcal{B}\varphi_1(t + \tau_n) - \mathcal{B}\psi_1(t)\|_{\mathbb{B}} &\leq \left\| \int_{-\infty}^t C(t - s)(\varphi_1(s + \tau_n) - \psi_1(s))ds \right\|_{\mathbb{B}} \\ &\leq \int_{-\infty}^t \|C(t - s)(\varphi_1(s + \tau_n) - \psi_1(s))\|_{\mathbb{B}} ds \\ &\leq \int_{-\infty}^t \rho(t - s)\|\varphi_1(s + \tau_n) - \psi_1(s)\|_{\mathbb{B}_{\frac{1}{2}}} ds \end{aligned}$$

which by Lebesgue Dominated Convergence Theorem yields

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{B}\varphi_1(t + \tau_n) - \mathfrak{B}\psi_1(t) \right\|_{\mathbb{E}} = 0.$$

Using similar arguments, we obtain

$$\lim_{n \rightarrow \infty} \left\| \mathfrak{B}\psi_1(t - \tau_n) - \mathfrak{B}\phi_1(t) \right\|_{\mathbb{E}} = 0.$$

For $r > 0$,

$$\begin{aligned} \frac{1}{2r} \int_{-r}^r \left\| \int_{-\infty}^t C(t-s)\varphi_2(s)ds \right\|_{\mathbb{E}} dt &\leq \frac{1}{2r} \int_{-r}^r \int_0^\infty \rho(s) \left\| \varphi_2(t-s) \right\|_{\mathbb{E}_{\frac{1}{2}}} ds dt \\ &\leq \int_0^\infty \rho(s) \left(\frac{1}{2r} \int_{-r}^r \left\| \varphi_2(t-s) \right\|_{\mathbb{E}_{\frac{1}{2}}} dt \right) ds. \end{aligned}$$

Now

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left\| \varphi_2(t-s) \right\|_{\mathbb{E}_{\frac{1}{2}}} dt = 0,$$

as $t \mapsto \varphi_2(t-s) \in PAP_0(\mathbb{E}_{\frac{1}{2}})$ for every $s \in \mathbb{R}$.

Therefore,

$$\lim_{r \rightarrow \infty} \frac{1}{2r} \int_{-r}^r \left\| \int_{-\infty}^t C(t-s)\varphi_2(s)ds \right\|_{\mathbb{E}} dt = 0$$

by using the Lebesgue Dominated Convergence Theorem.

Step 2. Clearly, the operator \mathcal{A} satisfies similar assumptions as (H.1)–(H.2) in the space $\mathbb{E}_{\frac{1}{2}}$. Indeed, for all $\varphi \in D(\mathcal{A})$, we have

$$\mathcal{A}\varphi = \sum_{n=1}^{\infty} \mathcal{A}_n P_n \varphi,$$

where

$$P_n := \begin{pmatrix} E_n & 0 \\ 0 & E_n \end{pmatrix} \text{ and } \mathcal{A}_n := \begin{pmatrix} 0 & 1 \\ -\lambda_n & -\lambda_n^\alpha \end{pmatrix}, \quad n \geq 1.$$

The characteristic equation for \mathcal{A}_n is given by

$$\rho^2 + 2\gamma\lambda_n^\alpha\rho + \lambda_n = 0,$$

from which we obtain its eigenvalues given by

$$\rho_1^n = \lambda_n^\alpha \left(-\gamma + \sqrt{\gamma^2 - \lambda_n^{1-2\alpha}} \right) \text{ and } \rho_2^n = \lambda_n^\alpha \left(-\gamma - \sqrt{\gamma^2 - \lambda_n^{1-2\alpha}} \right),$$

and hence $\sigma(\mathcal{A}_n) = \{\rho_1^n, \rho_2^n\}$.

Using Eq. (1.4) it follows that there exists $\omega > 0$ such that $\rho(\mathcal{A})$ contains the halfplane

$$S_\omega := \{\lambda \in \mathbb{C} : \Re \lambda \geq \omega\}.$$

Now since ρ_1^n and ρ_2^n are distinct and that each of them is of multiplicity one, then \mathcal{A}_n is diagonalizable. Further, it is not difficult to see that $\mathcal{A}_n = K_n^{-1} J_n K_n$, where J_n , K_n and K_n^{-1} are respectively given by

$$J_n = \begin{pmatrix} \rho_1^n & 0 \\ 0 & \rho_2^n \end{pmatrix}, \quad K_n = \begin{pmatrix} 1 & 1 \\ \rho_1^n & \rho_2^n \end{pmatrix},$$

and

$$K_n^{-1} = \frac{1}{\rho_1^n - \rho_2^n} \begin{pmatrix} -\rho_2^n & 1 \\ \rho_1^n & -1 \end{pmatrix}.$$

For $\lambda \in S_\omega$ and $\varphi \in \mathbb{E}_{\frac{1}{2}}$, one has

$$\begin{aligned} R(\lambda, \mathcal{A})\varphi &= \sum_{n=1}^{\infty} (\lambda - \mathcal{A}_n)^{-1} P_n \varphi \\ &= \sum_{n=1}^{\infty} K_n (\lambda - J_n)^{-1} K_n^{-1} P_n \varphi. \end{aligned}$$

Hence,

$$\begin{aligned} \|R(\lambda, \mathcal{A})\varphi\|_{\mathbb{E}_{\frac{1}{2}}}^2 &\leq \sum_{n=1}^{\infty} \|K_n (\lambda - J_n)^{-1} K_n^{-1}\|^2 \|P_n \varphi\|_{\mathbb{E}_{\frac{1}{2}}}^2 \\ &\leq \sum_{n=1}^{\infty} \|K_n\|^2 \|(\lambda - J_n)^{-1}\|^2 \|K_n^{-1}\|^2 \|P_n \varphi\|_{\mathbb{E}_{\frac{1}{2}}}^2. \end{aligned}$$

It is easy to see that there exist two constants $C_1, C_2 > 0$ such that

$$\|K_n\| \leq C_1 |\rho_1^n(t)|, \quad \|K_n^{-1}\| \leq \frac{C_2}{|\rho_1^n|} \quad \text{for all } n \geq 1.$$

Now

$$\begin{aligned} \|(\lambda - J_n)^{-1}\|^2 &= \left\| \begin{pmatrix} \frac{1}{\lambda - \rho_1^n} & 0 \\ 0 & \frac{1}{\lambda - \rho_2^n} \end{pmatrix} \right\|^2 \\ &\leq \frac{1}{|\lambda - \rho_1^n|^2} + \frac{1}{|\lambda - \rho_2^n|^2}. \end{aligned}$$

Define the function

$$\Theta(\lambda) := \frac{|\lambda|}{|\lambda - \rho_1^n(t)|}.$$

It is clear that Θ is continuous and bounded on S_ω . If we take

$$C_3 = \sup \left\{ \frac{|\lambda|}{|\lambda - \lambda_k^n|} : \lambda \in S_\omega, n \geq 1; k = 1, 2 \right\}$$

it follows that

$$\|(\lambda - J_n)^{-1}\| \leq \frac{C_3}{|\lambda|}, \quad \lambda \in S_\omega.$$

Therefore, one can find a constant $K \geq 1$ such

$$\|R(\lambda, \mathcal{A})\|_{B(\mathbb{E}_{\frac{1}{2}})} \leq \frac{K}{|\lambda|}, \quad \lambda \in S_\omega,$$

and hence the operator \mathcal{A} is sectorial on $\mathbb{E}_{\frac{1}{2}}$.

Since \mathcal{A} is sectorial in $\mathbb{E}_{\frac{1}{2}}$, then it generates an analytic semigroup $(\mathcal{T}(\tau))_{\tau \geq 0} := (e^{\tau \mathcal{A}})_{\tau \geq 0}$ on $\mathbb{E}_{\frac{1}{2}}$ given by

$$e^{\tau \mathcal{A}} \varphi = \sum_{n=0}^{\infty} K_n^{-1} P_n e^{\tau J_n} P_n K_n P_n \varphi.$$

First of all, note that the semigroup $(\mathcal{T}(\tau))_{\tau \geq 0}$ is hyperbolic as $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$. In other words, \mathcal{A} satisfies an assumption similar to (H.1).

Secondly, using the fact that \mathcal{A} is an operator of compact resolvent it follows that $\mathcal{T}(\tau)$ is compact for $\tau > 0$. On the other hand, we have

$$\|e^{\tau \mathcal{A}} \varphi\|_{\mathbb{B}_{\frac{1}{2}}} = \sum_{n=0}^{\infty} \|K_n^{-1} P_n\| \|e^{\tau J_n} P_n\| \|K_n P_n\| \|P_n \varphi\|_{\mathbb{B}_{\frac{1}{2}}},$$

with for each $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in \mathbb{B}_{\frac{1}{2}}$,

$$\begin{aligned} \|e^{\tau J_n} P_n \varphi\|_{\mathbb{B}_{\frac{1}{2}}}^2 &= \left\| \begin{pmatrix} e^{\rho_1^n \tau} E_n & 0 \\ 0 & e^{\rho_2^n \tau} E_n \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\|_{\mathbb{B}_{\frac{1}{2}}}^2 \\ &\leq \|e^{\rho_1^n \tau} E_n \varphi_1\|_{\frac{1}{2}}^2 + \|e^{\rho_2^n \tau} E_n \varphi_2\|^2 \\ &\leq e^{\Re(\rho_1^n) \tau} \|\varphi\|_{\mathbb{B}_{\frac{1}{2}}}^2. \end{aligned}$$

Using Eq. (1.4) it follows there exists $N' \geq 1$ such that

$$\|\mathcal{T}(\tau)\|_{B(\mathbb{B}_{\frac{1}{2}})} \leq N' e^{-\delta_0 \tau}, \quad \tau \geq 0,$$

and hence $\mathcal{T}(\tau)$ is exponentially stable, that is, \mathcal{A} satisfies an assumption similar to (H.2) in $\mathbb{B}_{\frac{1}{2}}$.

Step 3. The fact that F satisfies similar assumptions as (H.4) and (H.5) is clear. \square

5. EXAMPLE

In this section, we take $\alpha = \beta = \frac{1}{2}$. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with sufficiently smooth boundary $\partial\Omega$ and let $\mathbb{H} = L^2(\Omega)$ be the Hilbert space of all measurable functions $\varphi : \Omega \mapsto \mathbb{C}$ such that

$$\|\varphi\|_{L^2(\Omega)} = \left(\int_{\Omega} |\varphi(x)|^2 dx \right)^{1/2} < \infty.$$

Here, we study the existence of pseudo-almost automorphic solutions $\varphi(t, x)$ to a structurally damped plate-like system given by (see also Chen and Triggiani [13], Schnaubelt and Veraar [58], Triggiani [59]),

$$(5.1) \quad \begin{aligned} \frac{\partial^2 \varphi}{\partial t^2}(t, x) - 2\gamma \Delta \frac{\partial \varphi}{\partial t}(t, x) + \Delta^2 \varphi(t, x) &= \int_{-\infty}^t b(t-s) \varphi(s, x) ds + f(t, \varphi(t, x)) \\ &+ \eta \varphi(t, x), \quad (t, x) \in \mathbb{R} \times \Omega \end{aligned}$$

$$(5.2) \quad \Delta \varphi(t, x) = \varphi(t, x) = 0, \quad (t, x) \in \mathbb{R} \times \partial\Omega,$$

where $\gamma, \eta > 0$ are constants, $b : \mathbb{R} \mapsto [0, \infty)$ is a measurable function, the function $f : \mathbb{R} \times L^2(\Omega) \mapsto L^2(\Omega)$ is pseudo-almost automorphic in $t \in \mathbb{R}$ uniformly in the second variable, and Δ stands for the usual Laplace operator in the space variable x .

Setting

$$A\varphi = \Delta^2 \varphi \text{ for all } \varphi \in D(A) = D(\Delta^2) = \{\varphi \in H^4(\Omega) : \Delta \varphi = \varphi = 0 \text{ on } \partial\Omega\},$$

$$B\varphi = A^{\frac{1}{2}} \varphi = -\Delta \varphi, \quad \forall \varphi \in D(B) = H_0^1(\Omega) \cap H^2(\Omega),$$

$$C(t)\varphi = b(t)\varphi \text{ for all } \varphi \in D(C(t)) = D(\Delta),$$

and $f : \mathbb{R} \times (H_0^1(\Omega) \cap H^2(\Omega)) \mapsto L^2(\Omega)$, one can easily see that Eq. (1.2) is exactly the structurally damped plate-like system formulated in Eqs. (5.1)-(5.2).

Here $\mathbb{E}_{\frac{1}{2}} = D(A^{\frac{1}{2}}) \times L^2(\Omega) = (H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$ and it is equipped with the inner product defined by

$$\left(\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \right)_{\mathbb{E}_{\frac{1}{2}}} := \int_{\Omega} \Delta \varphi_1 \overline{\Delta \psi_1} dx + \int_{\Omega} \varphi_2 \overline{\psi_2} dx$$

for all $\varphi_1, \psi_1 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $\varphi_2, \psi_2 \in L^2(\Omega)$.

Similarly, $D(A^{\frac{1}{2}}) = H_0^1(\Omega) \cap H^2(\Omega)$ is equipped with the norm defined by

$$\|\varphi\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}\varphi\|_{L^2(\Omega)} := \left(\int_{\Omega} |\Delta \varphi|^2 dx \right)^{\frac{1}{2}}$$

for all $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$.

Clearly, $-A_{\eta} = -(\Delta^2 + \eta I)$ is a sectorial operator on $L^2(\Omega)$ and let $(T(t))_{t \geq 0}$ be the analytic semigroup associated with it. It is well-known that the semigroup $T(t)$ is not only compact for $t > 0$ but also is exponentially stable as

$$\|T(t)\| \leq e^{-\eta t}$$

for all $t \geq 0$.

Using the fact the Laplace operator Δ with domain $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$ is invertible in $L^2(\Omega)$ it follows that

$$\begin{aligned} \|\varphi\|_{L^2(\Omega)} &= \|\Delta^{-1} \Delta \varphi\|_{L^2(\Omega)} \\ &\leq \|\Delta^{-1}\|_{B(L^2(\Omega))} \cdot \|\Delta \varphi\|_{L^2(\Omega)} \\ &= \|\Delta^{-1}\|_{B(L^2(\Omega))} \cdot \|A^{\frac{1}{2}}\varphi\|_{L^2(\Omega)} \\ &= \|\Delta^{-1}\|_{B(L^2(\Omega))} \cdot \|\varphi\|_{\frac{1}{2}} \end{aligned}$$

for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$.

If $b \in L^1(\mathbb{R}, (0, \infty))$, then using the previous inequality it follows that

$$\begin{aligned} \|C(t)\Phi\|_{\mathbb{E}} &= b(t) \|\varphi\|_{L^2(\Omega)} \\ &\leq b(t) \|\Delta^{-1}\|_{B(L^2(\Omega))} \cdot \|\varphi\|_{\frac{1}{2}} \\ &\leq b(t) \|\Delta^{-1}\|_{B(L^2(\Omega))} \cdot \|\Phi\|_{\mathbb{E}_{\frac{1}{2}}} \end{aligned}$$

for all $\Phi = \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \in \mathbb{E}_{\frac{1}{2}}$ and $t \in \mathbb{R}$.

This setting requires the following assumptions,

(H.7) Eq. (1.4) holds.

(H.8) The function $b : \mathbb{R} \mapsto [0, \infty)$ belongs to $L^1(\mathbb{R}, (0, \infty))$ with

$$\int_{-\infty}^{\infty} b(s) ds \leq \frac{1}{2\|\Delta^{-1}\|_{B(L^2(\Omega))} d(\frac{1}{2})},$$

$$\text{where } d(\frac{1}{2}) := M(\frac{1}{2}) \sqrt{\frac{2\pi}{\eta}}.$$

In view of the above, it is clear that assumptions (H.1)–(H.2)–(H.3)–(H.6) are fulfilled. Therefore, using Corollary 4.8, we obtain the following theorem.

Theorem 5.1. *Under assumptions (H.4)–(H.5)–(H.7)–(H.8), then the system Eqs. (5.1)–(5.2) has at least one pseudo-almost automorphic mild solution.*

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